Exercise 5

Solve the differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = e^{2x}$$

Solution

This is a linear inhomogeneous ODE, so the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2y_c}{dx^2} - 4\frac{dy_c}{dx} + 5y_c = 0 \tag{1}$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad \frac{dy_c}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y_c}{dx^2} = r^2e^{rx}$$

Substitute these formulas into the ODE.

$$r^2e^{rx} - 4(re^{rx}) + 5(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 4r + 5 = 0$$

Solve for r.

$$r = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

$$r = \{2 - i, 2 + i\}$$

Two solutions to the ODE are $e^{(2-i)x}$ and $e^{(2+i)x}$. According to the principle of superposition, the general solution to equation (1) is a linear combination of these two.

$$\begin{aligned}
y_c(x) &= C_1e^{(2-i)x} + C_2e^{(2+i)x} \\
&= C_1e^{2x}e^{-ix} + C_2e^{2x}e^{ix} \\
&= e^{2x}(C_1e^{-ix} + C_2e^{ix}) \\
&= e^{2x}[C_1(\cos x - i\sin x) + C_2(\cos x + i\sin x)] \\
&= e^{2x}[(C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x] \\
&= e^{2x}(C_3\cos x + C_4\sin x)
\end{aligned}$$

C_3 and C_4 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2y_p}{dx^2} - 4\frac{dy_p}{dx} + 5y_p = e^{2x} \tag{3}$$

Since the inhomogeneous term is an exponential function, the particular solution is $y_p = Ae^{2x}$.

$$y_p = Ae^{2x} \quad \rightarrow \quad \frac{dy_p}{dx} = 2Ae^{2x} \quad \rightarrow \quad \frac{d^2y_p}{dx^2} = 4Ae^{2x}$$

Substitute these formulas into equation (3).

$$(4Ae^{2x}) - 4(2Ae^{2x}) + 5(Ae^{2x}) = e^{2x}$$

$$Ae^{2x} = e^{2x}$$

Match the coefficients to get an equation involving A.

$$A = 1$$

The particular solution is then

$$y_p = e^{2x}.$$

Therefore, the general solution to the original ODE is

$$\begin{aligned} y &= y_c + y_p \\ &= e^{2x}(C_3 \cos x + C_4 \sin x) + e^{2x}. \end{aligned}$$